the attainment of steady state (during "swinging in"). According to Peterlin [4], the time necessary approximately amounts to $4D/v^2$.

Comparison of experimental concentration profiles (ref. [1], Fig. 2) with values calculated from equations (4) and (5) shows relatively good conformity, at which, as in ref. [1], the time to reach the equilibrium has to be considered, as well as the fact that the precision of measurement diminishes with decreasing impulse count.

The occurrence of diffusion waves is of special interest for the co-operation of chemical reactions and diffusion. The present paper tries to extend the analogy between wave propagation and diffusion processes to the particular experimental case of diffusion with a moving boundary.

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SOME COMMENTS ON BECK'S SOLUTION OF THE INVERSE PROBLEM OF HEAT CONDUCTION THROUGH THE USE OF DUHAMEL'S THEOREM

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NOMENCLATURE

- a, radius of cylinder or sphere;
- erfc, complementary error function;
- E, depth of thermocouple below heated surface;
- F_r, least-square function;
- Fo, Fourier number, $\alpha t/L^2$ or $\alpha t/a^2$;
- J_0/J_1, Bessel function of the first kind of order zero and unity, respectively;
- k, thermal conductivity;
- L, thickness of slab;
- q_i, heat flux at time $t_i = i\Delta t$;
- $\bar{q}_i$, heat flux at time t_i determined by exactly matching the thermocouple data over r future times;
- $\bar{q}_{i+1}$, constant value of heat over the time interval t_i to t_{i+r} that minimizes the least-square error function F_r;
- r, radial coordinate, also the number of future temperatures used in inverse solution;
- T, temperature;
- T_E^i, computed temperature at time t_i and depth E below the heated surface;
- $\bar{T}_E^i$, value of T_E^i corresponding to $\bar{q}_i$;
- T_0, initial temperature;
- t, time;
- w^i, heat-flux weighting factor for a thermocouple at depth E, see equation (20);
- x, coordinate;
- Y^i, experimental thermocouple data at time $t_i = i\Delta t$.

Greek symbols

- ΔFo, differential Fourier number, $\alpha\Delta t/L^2$ or $\alpha\Delta t/a^2$;
- Δq_i, step change in heat flux, equation (2);
- Δt, time increment;
- $\Delta\phi^i$, $\phi^i-\phi^{i-1}$, $\phi^0=0$;
- α, thermal diffusivity;
- λ, dummy time variable;
- ν, eigenvalue in equations (4)–(6);
- ϕ, temperature response of a body initially at zero temperature and subjected to a unit step in heat flux, also termed sensitivity coefficient;
- ϕ_E^j, value of ϕ at depth E and time t_j;
- ψ, function representing decay of temperature profile if future heat-flux values are zero, see discussion following equation (9);
- ω_E^i, temperature weighting factor for a thermocouple at depth E; see equation (16).

INTRODUCTION

THE INVERSE problem of heat conduction is the determination of surface temperature and/or heat flux from an interior measurement of temperature. For those problems to which Duhamel's Theorem applies, Beck [1] introduced a technique for using measurements of future temperatures that allows for smaller computational steps than those allowed in the earlier technique of Stolz [2]. This note presents equations that permit an alternative physical interpretation of the process of using future temperature information, and gives additional insight into inverse heat conduction problems.

ANALYSIS

Starting with the 1-dim. form of Duhamel's Theorem, for a time-varying heat flux,

$$T(x,t) = T_0 + \int_0^t \phi(x,t-\lambda)\frac{\mathrm{d}q}{\mathrm{d}\lambda}\,\mathrm{d}\lambda + \sum_{i=0}^{n} \phi(x,t-\lambda_i)\Delta q_i \quad (1)$$

where

$$\Delta q_i = q_i - q_{i-1}, \quad q_0 \equiv 0$$

and where $\phi(x,t)$ is the temperature response of a body initially at zero temperature and subjected to a unit step in heat flux. The integral term in equation (1) allows for continuous variation of heat flux with time; the summation term accounts for discrete steps in heat flux. Many analytical solutions for bodies exposed to a step in heat flux are available in the literature; four solutions for common 1-dim. geometries are presented below.

Semi-infinite solid, flux at $x = 0$

$$\phi(x,t) = \frac{2}{k}\left[\left(\frac{\alpha t}{\pi}\right)^{1/2} e^{-x^2/4\alpha t} - \frac{x}{2}\,\mathrm{erfc}\,\frac{x}{2(\alpha t)^{1/2}}\right]. \quad (3)$$

Planar slab, thickness L, flux at x = 0, insulated inactive surface

$$\frac{\phi(x,t)k}{L} = \frac{\alpha t}{L^2} + \frac{1}{3} - \left(1 - \frac{x}{2L}\right)\frac{x}{L} - 2\sum_{n=1}^{\infty}\frac{1}{v_n^2}e^{-(\alpha t/L^2)v_n^2}\cos\left(v_n\frac{x}{L}\right), \quad v_n = n\pi \quad (4)$$

Solid cylinder, radius a, flux at r = a

$$\frac{\phi(r,t)k}{a} = \frac{2\alpha t}{a^2} + \frac{r^2}{2a^2} - \frac{1}{4} - 2\sum_{n=1}^{\infty} e^{-v_n^2(\alpha t/a^2)}\frac{J_0(v_n r/a)}{v_n^2 J_0(v_n)},$$

$$J_1(v_n) = 0, \quad \text{positive roots.} \quad (5)$$

Solid sphere, radius a, flux at r = a

$$\frac{\phi(r,t)k}{a} = \frac{3\alpha t}{a^2} + \frac{1}{10}\left(\frac{5r^2}{a^2} - 3\right) - 2\frac{a}{r}\sum_{n=1}^{\infty}\frac{\sin(v_n r/a)}{v_n^2 \sin v_n}e^{-v_n^2\alpha t/a^2} \quad \tan v_n = v_n. \quad (6)$$

For each of the above finite-body cases, the infinite-series term approaches zero for large values of time, and the response is linear with time. For large $Fo = \alpha t/L^2$ (or $\alpha t/a^2$), the rate at which the temperature of the cylinder rises is twice as fast as the rate for the plate, and the rate for the sphere is three times as fast as the rate for the plate at a given heat flux.

Assuming that the temperature profile is known at time $t_M = M\Delta t$, the temperature at any arbitrary depth E below the heated surface can be calculated. If the heat flux consists of a series of steps, the integral term in equation (1) is zero, and the temperature at location E can be calculated from equation (1) as follows:

$$T_E^{M+1} = T_0 + \sum_{i=1}^{M} q_i \Delta\phi_E^{M-i+2} + q_{M+1}\Delta\phi_E^1 \quad (7a)$$

$$T_E^{M+2} = T_0 + \sum_{i=1}^{M} q_i \Delta\phi_E^{M-i+3} + q_{M+1}\Delta\phi_E^2 + q_{M+2}\Delta\phi_E^1 \quad (7b)$$

$$T_E^{M+j} = T_0 + \sum_{i=1}^{M} q_i \Delta\phi_E^{M-i+j+1} + \sum_{i=1}^{j} q_{M+i}\Delta\phi_E^{j-i+1} \quad (7c)$$

with $\Delta\phi^i = \phi^i - \phi^{i-1}$, $\phi^0 = 0$. It will be convenient to define a temperature $\psi_E^{M,j}$ as

$$\psi_E^{M,j} = T_0 + \sum_{i=1}^{M} q_i \Delta\phi_E^{M-i+j+1} \quad (8)$$

so that equation (7c) can be written as

$$T_E^{M+j} = \psi_E^{M,j} + \sum_{i=1}^{j} q_{M+i}\Delta\phi_E^{j-i+1}. \quad (9)$$

In physical terms, $\psi_{(x)}^{M,0}$ is the temperature profile at time $M\Delta t$ corresponding to the sequence of heat-flux values $q_1, q_2, \ldots, q_M$; $\psi_{(x)}^{M,j}$ represents the decay of this 'initial' temperature profile if the heat flux is set to zero for the j future time steps ($q_{M+1} = q_{M+2} = \ldots = q_{M+j} = 0$) Because we are usually concerned with the temperature at a location E, $\psi_E^{M,j}$ represents the decay of a thermocouple at location E when all future heat flux terms are set to zero.

The Beck procedure is to determine the heat flux that minimizes the least-square error between r future temperatures as computed from equation (9) and the experimental thermocouple data. The least-square error is

$$F_r = \sum_{j=1}^{r}(T_E^{M+j} - Y^{M+j})^2. \quad (10)$$

Minimizing F_r with respect to q_{M+k} gives

$$\frac{\partial F_r}{\partial q_{M+k}} = 2\sum_{j=1}^{r}(T_E^{M+j} - Y^{M+j})\frac{\partial T_E^{M+j}}{\partial q_{M+k}} = 0,$$

$$k = 1, 2, \ldots, r. \quad (11)$$

To evaluate the sensitivity coefficients $\partial T_E^{M+j}/\partial q_{M+k}$, Beck made the temporary assumption that $q_{M+1} = q_{M+2} = q_{M+r} = \bar{q}_{M+1}$. Under this assumption,

$$\frac{\partial T_E^{M+j}}{\partial q_{M+k}} = \frac{\partial T_E^{M+j}}{\partial \bar{q}_{M+1}} = \phi_E^j. \quad (12)$$

Because of the temporary assumption of a constant heat flux, equation (11) reduces to a single equation for the single unknown heat flux $\bar{q}_{M+1}$.

$$\bar{q}_{M+1} = \frac{\sum_{j=1}^{r}\phi_E^j(Y^{M+j} - \psi_E^{M,j})}{\sum_{j=1}^{r}(\phi_E^j)^2}. \quad (13)$$

Equation (13) is similar to the result first developed by Beck. For $r = 1$, the Beck method reduces to the Stolz method in which the thermocouple data are matched exactly. The sensitivity coefficients ϕ_E^j must be calculated from equations (3)–(6) or other appropriate relationships before equation (13) can be applied.

An alternative and possibly more enlightening form of equation (13) can be developed. After the heat flux $\bar{q}_{M+1}$ has been calculated, the direct problem is then solved, yielding a temperature at thermocouple depth E. This computed thermocouple temperature corresponding to $\bar{q}_{M+1}$ will be denoted by $\bar{T}_E^{M+1}$. If all values of $\bar{T}_E^j$, $j = 1, 2, \ldots$ are known, then the Stolz procedure could conceivably be used to determine the heat flux that exactly matches the thermocouple data. From equation (9), the heat flux $\bar{q}_{M+1}$ and computed thermocouple temperature T^{M+1} are related through a single linear relationship.

$$\bar{T}_E^{M+1} = \psi_E^{M,1} + \bar{q}_{M+1}\phi_E^1. \quad (14)$$

Substituting equation (13) into equation (14) gives

$$\bar{T}_E^{M+1} - \psi_E^{M,1} = \frac{\phi_E^1 \sum_{j=1}^{r}\phi_E^j(Y^{M+j} - \psi_E^{M,j})}{\sum_{k=1}^{r}(\phi_E^k)^2}$$

or

$$\bar{T}_E^{M+1} - \psi_E^{M,1} = \sum_{j=1}^{r}\omega_E^j(Y^{M+j} - \psi_E^{M,j}) \quad (15)$$

where ω_E^j is a temperature weighting factor defined by

$$\omega_E^j = \frac{\phi_E^1\phi_E^j}{\sum_{k=1}^{r}(\phi_E^k)^2}. \quad (16)$$

Equation (15) indicates that the inverse solution obtained by using data from r future times is the same as the solution obtained by averaging the experimental thermocouple data in a certain way and exactly matching the averaged thermocouple data ($\bar{T}_E^{M+1}$). The weighting factors ω_E^j depend only on the sensitivity coefficients, and can be calculated before any inverse calculations. Note, however, it is not possible to first 'smooth' the entire thermocouple data set and then use the Stolz procedure to exactly match the smoothed data because the $\psi_E^{M,j}$ functions must be continuously re-calculated as the inverse solution evolves.

It is possible to convert the temperature-averaging equation (15) into a heat-flux-averaging equation. Let $\tilde{q}_{M+i}$ represent the heat flux at time $(M+i)\Delta t$ obtained by exactly matching the sequence of thermocouple data points Y^{M+1}, $Y^{M+2}, \ldots, Y^{M+i}$ and starting from the temperature profile corresponding to $\tilde{q}_M$. Applying the discrete form of Duhamel's theorem, equation (9), to this situation gives

$$Y^{M+j} - \psi_E^{M,j} = \sum_{i=1}^{j} \tilde{q}_{M+i}\Delta\phi_E^{j-i+1} \qquad (17)$$

and substituting equation (17) into equation (13) gives

$$\tilde{q}_{M+1} = \frac{\sum_{j=1}^{r} \phi_E^j \sum_{i=1}^{j} \tilde{q}_{M+i}\Delta\phi_E^{j-i+1}}{\sum_{j=1}^{r} (\phi_E^j)^2}. \qquad (18)$$

If the summation terms of $\tilde{q}_{M+1}$ are expanded and its coefficients are collected, equation (18) can be written as

$$\tilde{q}_{M+1} = \sum_{i=1}^{r} w_E^i \tilde{q}_{M+i} \qquad (19)$$

Table 1. Asymptotic values of temperature and heat-flux weighting coefficients ($\alpha\Delta t/L^2$ large)

r	ω^1	ω^2	ω^3	ω^4	ω^5
1	1.0	—	—	—	—
2	0.2	0.4	—	—	—
3	0.0714	0.1429	0.2143	—	—
4	0.0333	0.0667	0.1000	0.1333	—
5	0.0182	0.0364	0.0545	0.0727	0.0909
r	w^1	w^2	w^3	w^4	w^5
1	1.0	—	—	—	—
2	0.6	0.4	—	—	—
3	0.4286	0.3571	0.2143	—	—
4	0.3333	0.3000	0.2333	0.1333	—
5	0.2727	0.2545	0.2182	0.1636	0.0909

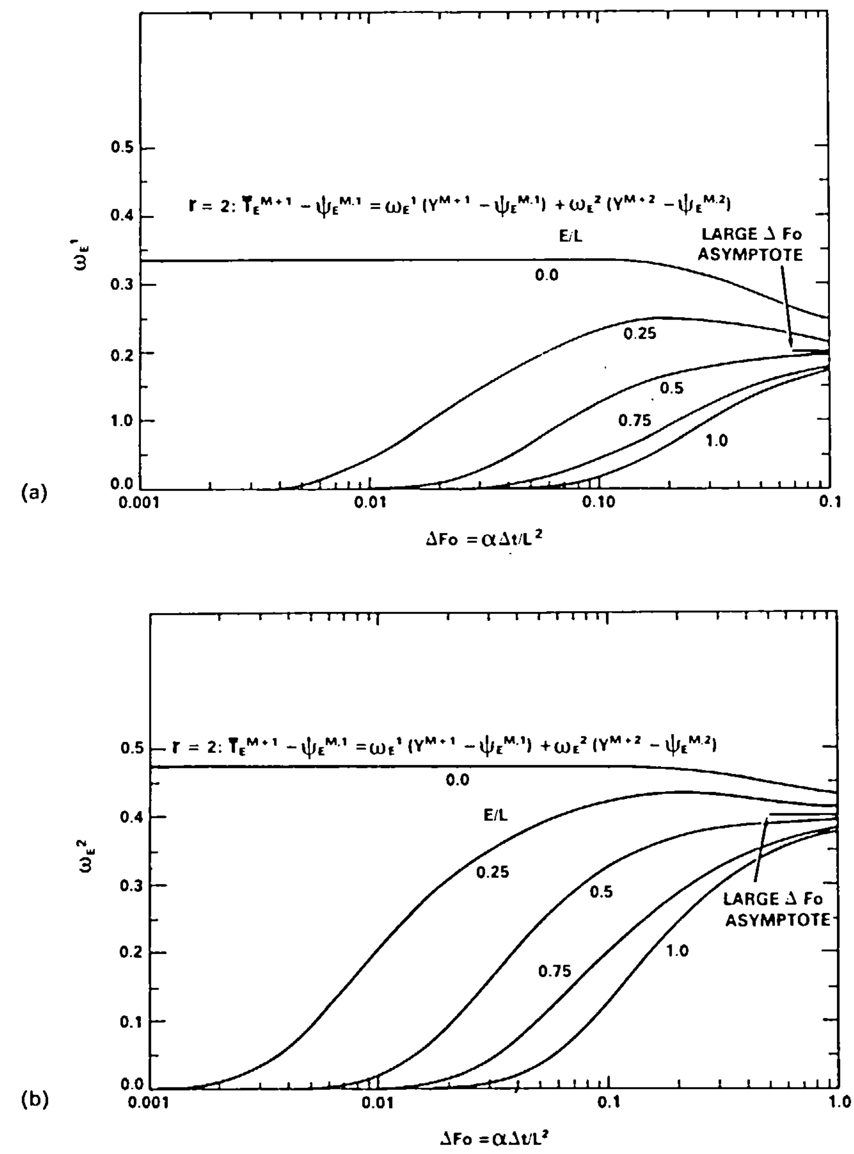

FIG. 1. Variation of temperature weighting coefficients with $\alpha\Delta t/L^2$ for $r = 2$, planar geometry, insulation inactive surface: (a) ω_E^1; (b) ω_E^2.

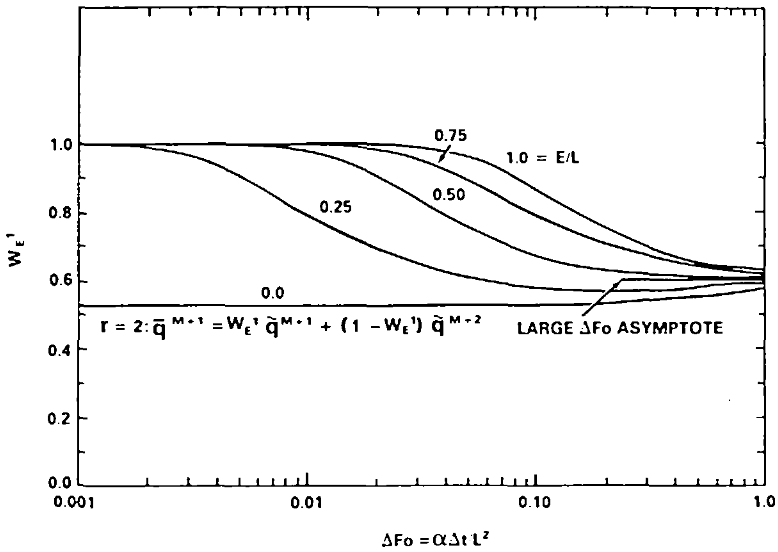

FIG. 2. Variation of heat flux weighting coefficient with $\alpha\Delta t/L^2$ for $r = 2$, planar geometry, insulated inactive surface.

where the heat-flux weighting factors w_E^i are given by

$$w_E^i = \frac{\sum_{k=i}^{r} \phi_E^k \Delta\phi_E^{k-i+1}}{\sum_{k=1}^{r} (\phi_E^k)^2}, \quad \Delta\phi^j = \phi^j - \phi^{j-1}. \tag{20}$$

Equation (19) indicates that the heat flux determined from the Beck procedure is a weighted average of the r heat flux values $\tilde{q}_{M+i}$, $i = 1,2,\ldots,r$ determined by exactly matching the r future thermocouple data points. It can be shown that the heat-flux weighting factors sum to unity

$$\sum_{i=1}^{r} w_E^i = 1.0. \tag{21}$$

The heat-flux weighting factors w_E^i depend only on the sensitivity coefficients and require calculation only once for problems in which Duhamel's Theorem is valid.

Both the temperature and heat-flux weighting factors take on very simple forms for 1-dim. geometries when $\alpha\Delta t/L^2$ or $\alpha\Delta t/a^2$ is large. For this condition, the series terms in equations (4)–(6) can be ignored. The sensitivity coefficients vary linearly with time and are approximately independent of thermocouple depth. It can be demonstrated that

$$\left.\begin{aligned} \omega_E^{M+i} &\approx \frac{i}{\sum_{k=1}^{r} k^2} \\ w_E^{M+i} &\approx \frac{\sum_{k=i}^{r} k}{\sum_{k=1}^{r} k^2} \end{aligned}\right\} \quad \text{large } \frac{\alpha\Delta t}{L^2} \text{ limit.} \tag{22}$$

Note that planar, cylindrical, and spherical geometries all have the same weighting factors for the limiting case of large $\alpha\Delta t/L^2$ and are independent of thermocouple depth. Table 1 tabulates the results of equation (22) for $r = 1$–5. When all values of w_E^i, $i = 1,2,\ldots,r$, are considered, w_E^1 will always be the largest.

Figures 1 and 2 present the temperature and heat-flux weighting coefficients as a function of $\alpha\Delta t/L^2$ for planar geometry, insulated inactive surface, and $r = 2$. For small $\alpha\Delta t/L^2$, the influence of the future data becomes less and less. This gives some insight into why there are potential stability problems for small values of $\alpha\Delta t/L^2$. Smaller values of $\alpha\Delta t/L^2$ may require large values of r for stability. For a given value of $\alpha\Delta t/L^2$, increasing the thermocouple depth E/L increases w_E^1; this implies that future information is given a smaller weighting.

SUMMARY

An alternative interpretation of Beck's integral solution of the inverse problem of heat conduction has been presented. The use of future temperature information and the minimization of the least-square error between computed and experimental thermocouple data can be interpreted as (1) matching the thermocouple data exactly over r future times and (2) averaging the resulting heat flux values $\tilde{q}_{M+i}$ according to equation (19). Limiting values of temperature and heat-flux weighting coefficients for large values of $\alpha\Delta t/L^2$ are presented. Typical values of ω and w are presented for $r = 2$ and a planar geometry.

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